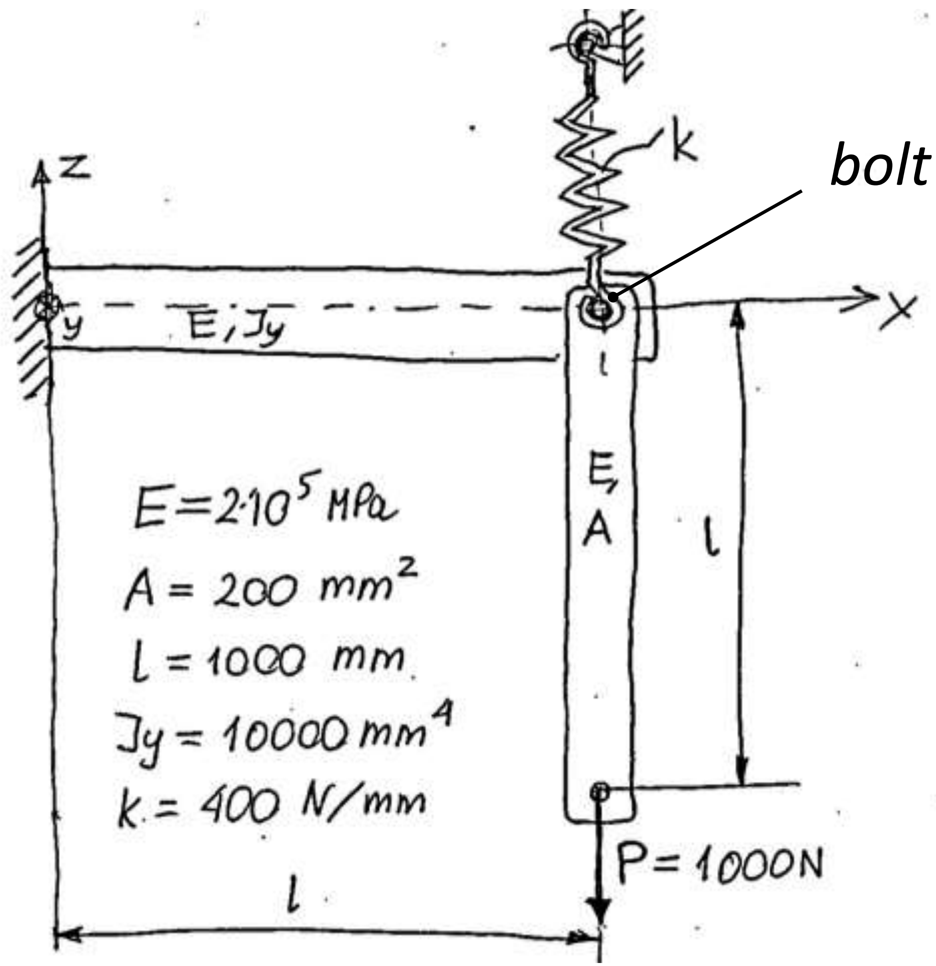
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

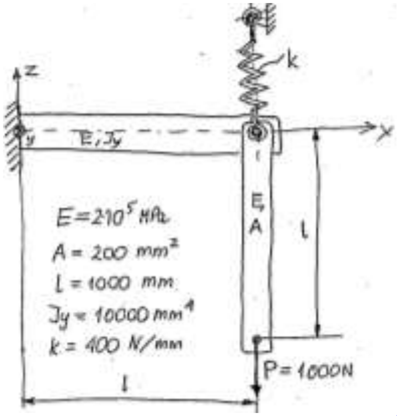
Finite element method (FEM1)

Lecture 9C. Beam, spring & bar assembly - example

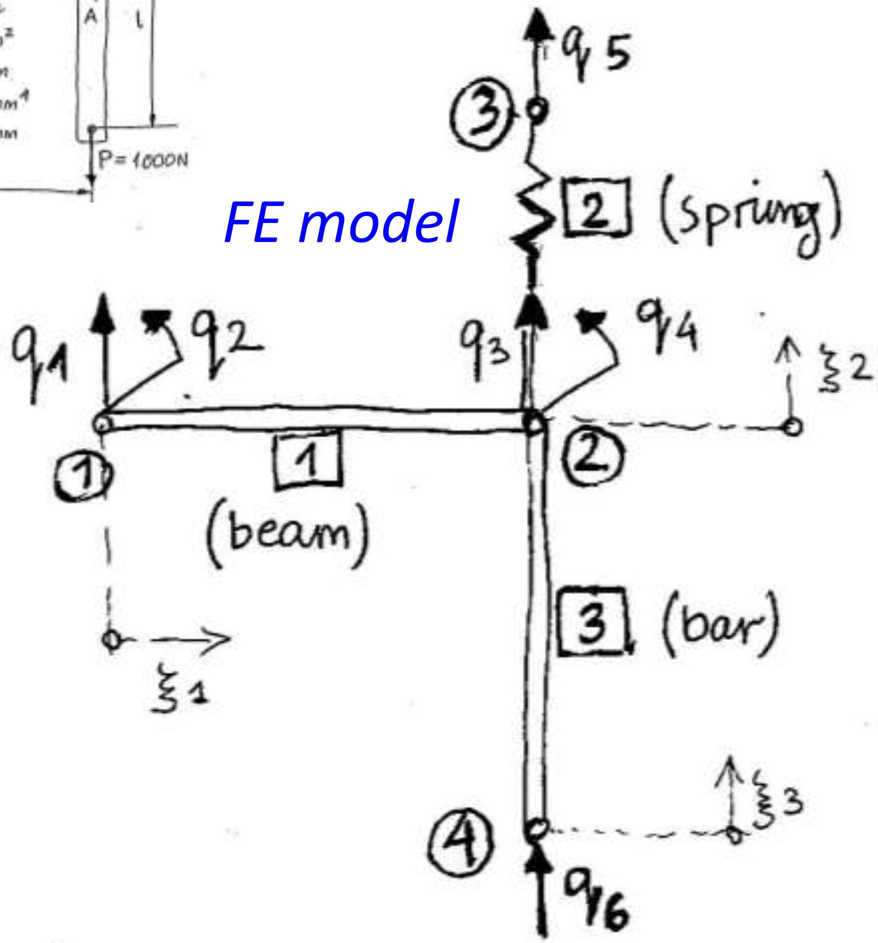
05.2025

Example: BUILD A FINITE ELEMENT MODEL OF THE STRUCTURE CONSISTING OF A BEAM, A BAR AND A SPRING. FIND UNKNOWN DISPLACEMENTS AND REACTIONS AND CHECK EQUILIBRIUM.



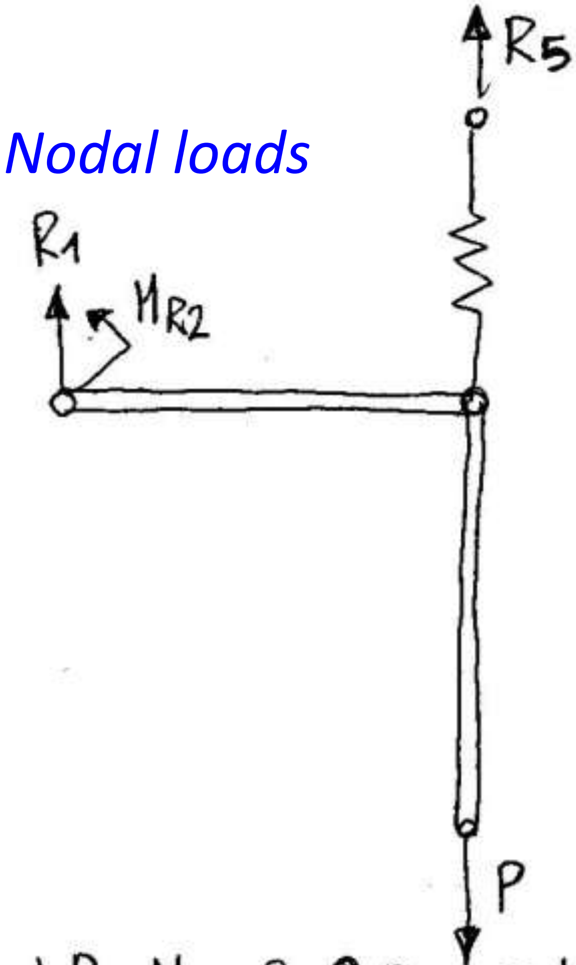


FE model



$$\lfloor q \rfloor_{1\times 6} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6 \rfloor$$

Nodal loads



$$\lfloor F \rfloor_{1\times 6} = \lfloor R_1, M_{R2}, 0, 0, R_5, -P \rfloor$$

element $\boxed{1}$: $\underset{1\times 4}{\lfloor q \rfloor_1} = \lfloor q_1, q_2, q_3, q_4 \rfloor_1$

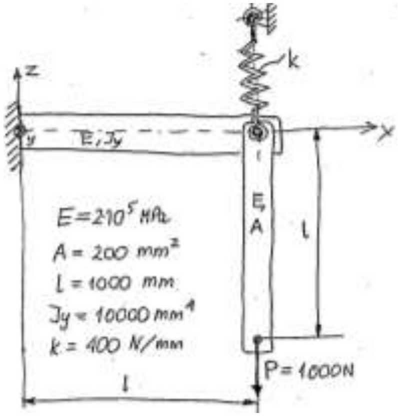
$$\underset{4\times 4}{[k]_1} = \frac{2EJ_y}{l^3} \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \quad ; \quad \underset{6\times 6}{[k]_1^*} = \left[\begin{array}{cccc|cc} \multicolumn{4}{c|}{\underset{4\times 4}{[k]_1}} & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

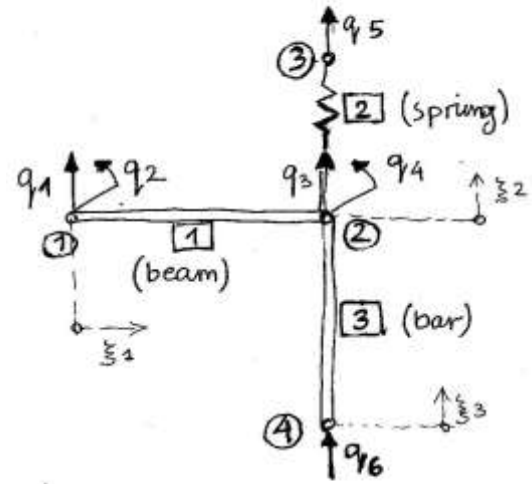
element $\boxed{2}$: $\underset{1\times 2}{\lfloor q \rfloor_2} = \lfloor q_3, q_5 \rfloor_2$

$$\underset{2\times 2}{[k]_2} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad ; \quad \underset{6\times 6}{[k]_2^*} = \begin{array}{c} \begin{array}{cccccc} & & (3) & & (5) & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{k} & 0 & \boxed{-k} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-k} & 0 & \boxed{k} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ (3) \\ \\ (5) \\ \\ \end{array}$$

element $\boxed{3}$: $\underset{1\times 2}{\lfloor q \rfloor_3} = \lfloor q_6, q_3 \rfloor_3$

$$\underset{2\times 2}{[k]_3} = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad ; \quad \underset{6\times 6}{[k]_3^*} = \begin{array}{c} \begin{array}{cccccc} & & (3) & & & (6) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{\frac{EA}{l}} & 0 & 0 & \boxed{-\frac{EA}{l}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-\frac{EA}{l}} & 0 & 0 & \boxed{\frac{EA}{l}} \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ (3) \\ \\ \\ (6) \end{array}$$





$$
[K] = \sum_{e=1}^{3} [k]_e^{*} = \left[\begin{array}{cccc|cc}
 & & & & 0 & 0 \\
 & [K]_1 & & & 0 & 0 \\
 & & \frac{12EJ_y}{l^3} + k + \frac{EA}{l} & & -k & -\frac{EA}{l} \\
 & & & & 0 & 0 \\
\hline
0 & 0 & -k & 0 & k & 0 \\
0 & 0 & -\frac{EA}{l} & 0 & 0 & \frac{EA}{l}
\end{array}\right]
$$

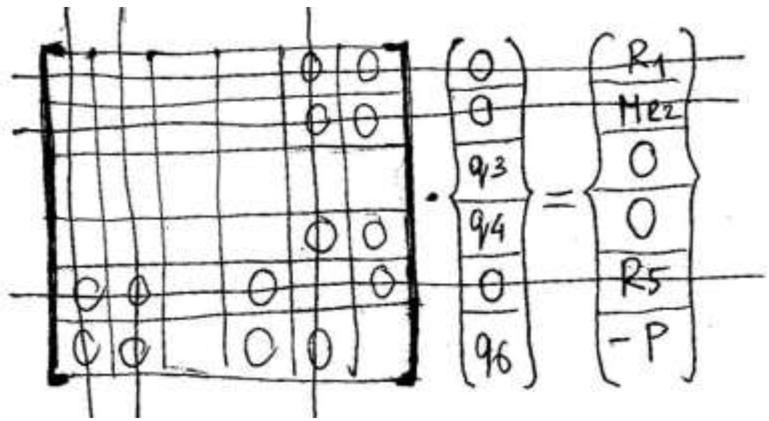
$$\underset{6\times6}{[K]}\cdot\underset{6\times1}{\{q\}}=\underset{6\times1}{\{F\}}$$

\+ boundary conditions

$q_1 = 0$

$q_2 = 0$

$q_5 = 0$



$$\rightarrow \underset{3\times3}{[K]}\cdot\underset{3\times1}{\{q\}}=\underset{3\times1}{\{F\}}$$

$$\begin{array}{c} \mathrm{I} \\[4ex] \mathrm{II} \\[4ex] \mathrm{III} \end{array}
\begin{bmatrix} \frac{12EJ_y}{l^3}+k+\frac{EA}{L} & -\frac{6EJ_y}{l^2} & -\frac{EA}{L} \\ -\frac{6EJ_y}{l^2} & \frac{4EJ_y}{L} & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} \end{bmatrix}
\cdot \begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix}$$

$$
\begin{array}{c}
\mathrm{I} \\ \\ \mathrm{II} \\ \\ \mathrm{III}
\end{array}
\begin{bmatrix}
\frac{12EJ_y}{l^3}+k+\frac{EA}{l} & -\frac{6EJ_y}{l^2} & -\frac{EA}{l} \\
-\frac{6EJ_y}{l^2} & \frac{4EJ_y}{l} & 0 \\
-\frac{EA}{l} & 0 & \frac{EA}{l}
\end{bmatrix}
\cdot
\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix}
$$

Eq. III) $-\frac{EA}{l}\cdot q_3 + 0\cdot q_4 + \frac{EA}{l} q_6 = -P$

$$q_6 = q_3 - \frac{PL}{EA}$$

Eq II) $-\frac{6EJ_y}{l^2} q_3 + \frac{4EJ_y}{l}\cdot q_4 + 0\cdot q_6 = 0$

$$q_4 = \frac{3}{2l}\cdot q_3$$

Eq I) $\left(\frac{12EJ_y}{l^3}+k+\frac{EA}{l}\right)\cdot q_3 - \frac{6EJ_y}{l^2}\cdot q_4 - \frac{EA}{l}\cdot q_6 = 0$

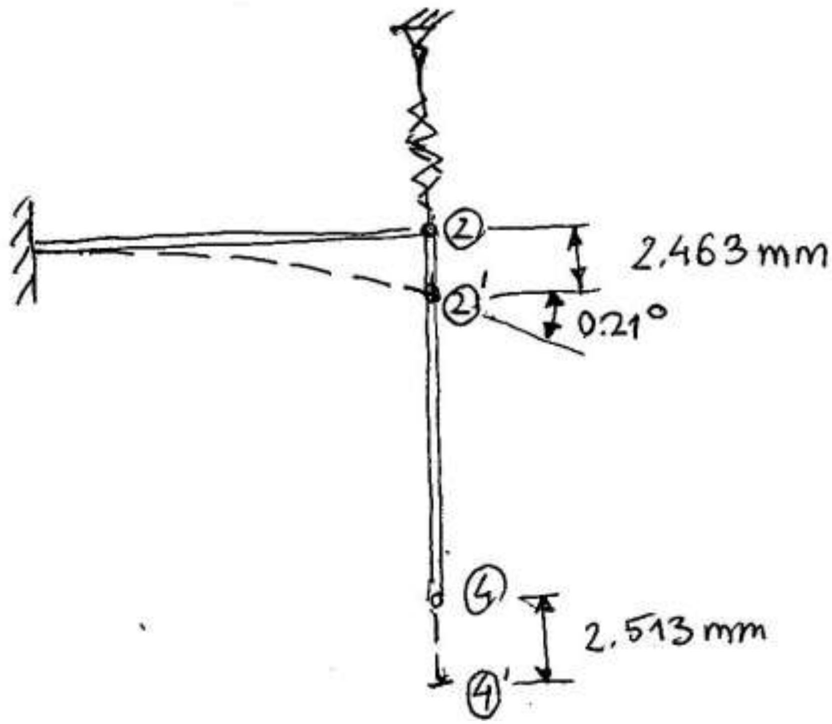
$$\left(\frac{12EJ_y}{l^3}+k+\frac{EA}{l} - 9\frac{EJ_y}{l^3} - \frac{EA}{l}\right) q_3 = -\frac{PL}{EA}\cdot\frac{EA}{l} = -P$$

Solution:

$$q_3 = -\frac{P}{\frac{3EJ_y}{l^3}+k} = -2.463\,\text{mm}$$

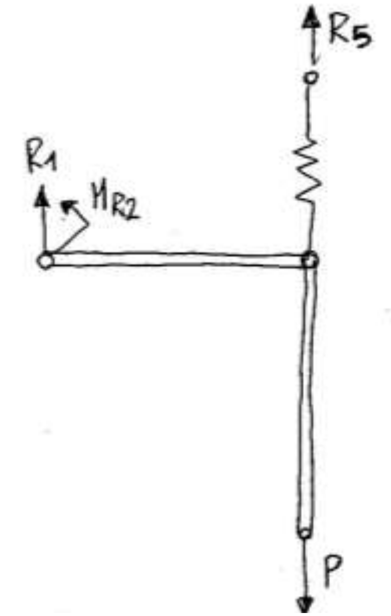
$$q_4 = \frac{3}{2l}\cdot q_3 = -0.00365\,\text{rad} = -0.21^\circ$$

$$q_6 = q_3 - \frac{PL}{EA} = -2.463\,\text{mm} - 0.05\,\text{mm} = -2.513\,\text{mm}$$



Reactions:

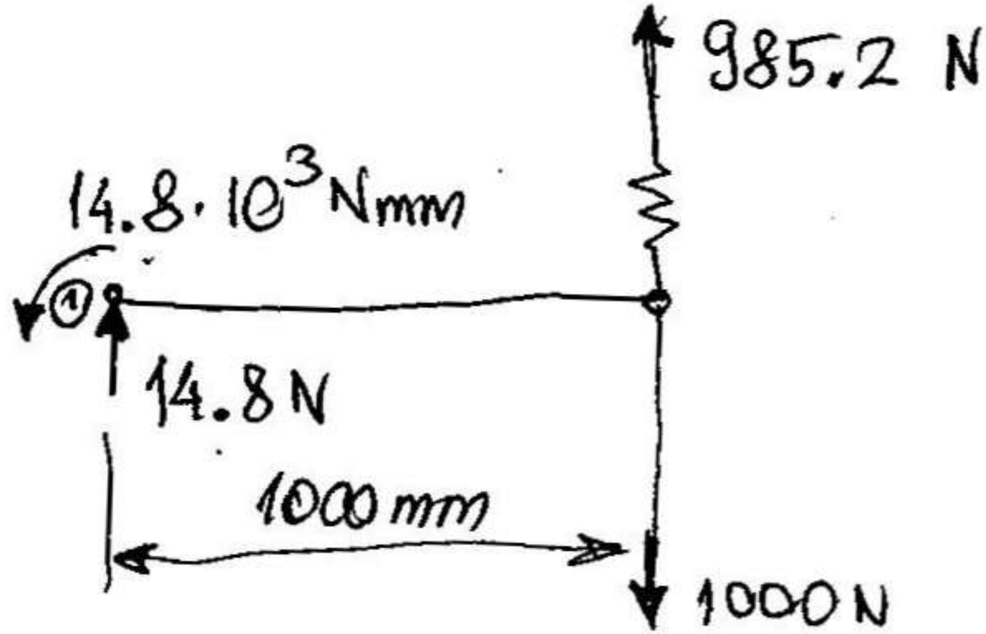
$$[K]_{6\times6}\cdot\{q\}_{6\times1}=\{F\}_{6\times1}$$



$$\begin{cases} -\dfrac{12EJ_y}{l^3}\cdot q_3+\dfrac{6EJ_y}{l^3}q_4+0\cdot q_6=R_1 \\ -6\dfrac{EJ_y}{l^2}q_3+\dfrac{2EJ_y}{l}q_4+0\cdot q_6=M_{R2} \\ -k\cdot q_3+0\cdot q_4+0\cdot q_6=R_5 \end{cases}$$

$R_1 = 14.8\,N$, $M_{R2} = 14.8\cdot10^3\,Nmm$, $R_5 = 985.2\,N$

Equilibrium:



$$\sum F_z = 0;$$

$$14.8\,N + 985.2\,N - 1000\,N = 0\,N$$

$$\sum M_y^{①} = 0 \quad +\circlearrowleft$$

$$14.8 \cdot 10^3\,Nmm - 1000\,N \cdot 1000\,mm + 985.2\,N \cdot 1000\,mm = 0\,Nmm$$

(satisfied)